Possible applications of knots in Computer Music and NIMEs

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ABSTRACT

In this paper, we endeavor to examine specific phenomena - knots - which are subjects of study in various scientific disciplines, including a particular field within mathematics. Our main goal is to explore the possibilities that knots open for computer music, particularly for NIME researchers. We aim to achieve this goal by analyzing four aspects of knots: topology, geometry, physics, and semantics. Subsequently, we apply these aspects to areas of computer music with examples, some of which are accompanied by proof-of-concept models, while others remain purely conceptual, awaiting further practical research. Although most cases draw inspiration from mathematical Knot Theory, not all strictly adhere to its conditions.

Author Keywords

computer music, knots, NIME, topology

CCS Concepts

•**Applied computing** → **Sound and music computing;** Performing arts;

1. INTRODUCTION

Knots, visually complex and intriguing [38], embody configurability and captivate curiosity [11, 38]. Their shapes, derived from bodily movements, act as operational trajectories, motivating precise actions.

Physically created knots, like the Slip Knot and Bowline Knot, share a source and may look alike, but differ fundamentally in functionality. Knots are configurable machines.

Becoming more recognized [35], they are subjects of interdisciplinary studies. They find applications in physics, mechanics, cultural studies, biology, etc. [1, 19]. Collaborations across zoology, computer science, material research, and robotics explore the unique capabilities of knotted living organisms [10, 29].

Mathematical studies of knots impact modern disciplines [1, 35], inspiring the abstract field of Knot Theory (within Topology). The intersection of Computer Music and knots exists, but still is a new area [12, 13, 14, 31, 5], presenting an intriguing avenue for exploration.

The scope of this paper precludes a comprehensive overview of Knot Theory; however, recommended sources are available [1, 35, 23, 33]. Nevertheless, we provide a brief outline of the concepts, which are essential for our exploration.

1.1 Mathematical Knots

Knot Theory centers on the abstract notion of a mathematical knot, represented as a simple closed curve in three-dimensional space [23]. The most basic is the "trivial knot" or "unknot," seen as a circle in 3D space. All mathematical knots are tied (tangled) on a looped curve. In most simple words Knot Theory claims that those knots, which can not be untangled to the state of visible circle - might be significantly different between each other. Knot Theory provides tools to distinct and classify those different knots.

1.2 Topological vs Geometric Knots

Knot Theory distinguishes between topological and geometric knots. Topological knots consider equivalence under continuous deformations, while geometric knots account for spatial embeddings and geometric features [1, 35].

1.3 Crossings

Knots can be identified by the number of crossings, where the curve self-intersects. Knot tabulations are based on this characteristic. While a mathematical knot can have any number of crossings, reducing it to its simplest form yields a specific crossing number. Knots with the same number of crossings may differ, but the variety is likely finite.

1.4 Knot Projections in 2D

Many knot problems utilize 2D projections on a plane, represented as diagrams or shadows. Diagrams code crossing information, while shadows lack such details [18, 27].

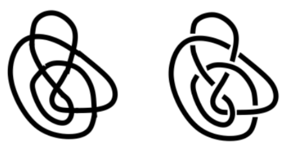

Figure 1: Shadow (left) and a Diagram (right) (from [33])

Sometimes terms "projection" and "shadow" are mixed.

NIME'24, 4–6 September, Utrecht, The Netherlands.

1.5 Knot Equivalence Problem

The equivalence problem, determining how to differentiate between different knots, relies on twp main tools: Reidemeister Moves and Knot Invariants. Reidemeister Moves involve three basic operations on knot diagrams. Every tangled knot can be transformed to it's most basic form by sequential use of the moves (Figure 2). Which downgrades the number of crossings to minimum.

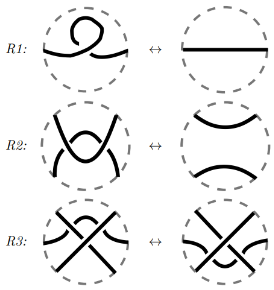

Figure 2: Reidemeister Moves (from [33])

Knot Invariants are unchanged properties used for distinction. The list of knot invariants is very long and it keeps growing. Some of them are based on topological and geometric characteristics (e.g. tricolorability, p-colorability, etc.). Others are coded in expression like polynomials (e.g. Alexander Polynomial, Jones Polynomial, HOMFLY Polynomial, etc.). "Minimal number of crossings" - is actually one of the invariants [33].

1.6 Types of Knots

Knot Theory classifies knots into various types. For example, Prime Knots are fundamental basis for all other knots, and Torus Knots are those that can be embedded on the surface of a torus [35, 1, 4]. Torus Knots are identified with two coprime integers (p, q). An example of a simple non-trivial torus knot is the (2,3)-torus knot, commonly known as the trefoil (Figure 3).

Figure 3: Diagram of a trefoil knot (from [33])

1.7 Braids

Knot Theory extends beyond knots to include Braids, which are arrangements of parallel strands. Braid Theory, related to Knot Theory, explores connections between knots and braids [33, 1].

A closure of specific braids can be used to represent any knot (Figure 4).

1.8 Computing a Knot

352,152,252 distinct non-trivial prime knots have been tabulated by year 2020 [4]. Knots can be computed using Fourier parametrization or specific parametric equations, such as those for Torus Knots [36, 2]. Below is the formal inscription for a trefoil knot:

$$Trefoil\ (a\ simple\ Torus\ knot):\ 3_1 knot(3,2)$$

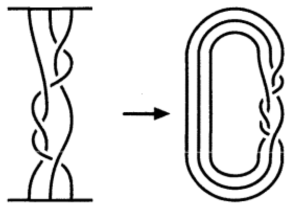

Figure 4: A braid (left) closes into a knot (right) (from [1])

where "3" is the minimum number of crossings, lower-case "1" is the order index of the knot with 3 crossings in the standard knot tabulation (which is beyond our scope) and "(3, 2)" are the "(p, q)" integers.

A set of parametric equations for Torus knots:

$$x = r\cos(pt)$$
$$y = r\sin(pt) \qquad (1)$$
$$z = -\sin(qt)$$

where

$$[r = \cos(qt) + 2] \text{ and } [0 < t < 2\pi]$$

After assigning "(p, q)" values, we can see these equations plotted in 3D in a form of a trefoil knot:

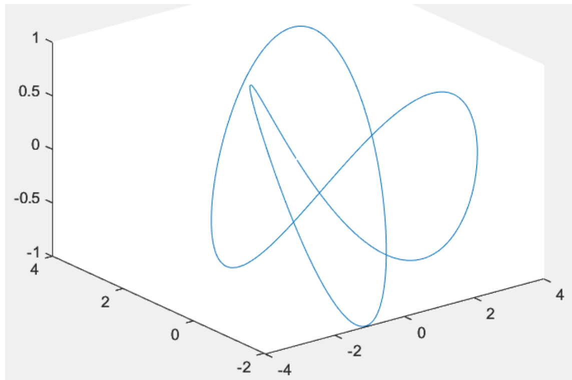

Figure 5: Realisation of a trefoil (in MatLab)

2. ON POSSIBLE APPLICATIONS

2.1 What do Knots provide to us?

The limited introduction to the Knot Theory given above can be summarized into practical outcomes, according to two domains: geometry and topology. We add to them another two, which are not investigated in the Knot Theory, but useful for our purpose: physics and semantics. We result in four aspects:

Geometrically: Knots provide to us a possibility to compute various types of tangled trajectories in 3D (or 2D) spaces. They may look chaotic, but in fact would be strictly determined. Some of them (like Torus Knots) would have symmetry and regularity.

Topologically: Knots provide to us a tool to track some core unchangeable characteristics under continuous and potentially infinite deformations. This would be true to anything, which could be coded as a deformation of a looped curve in 3D (or 2D in case of projections).

Physically: Knots form potentially infinite number of different 3D structures, which can be created from a basic source, equal in its features to a single filament. These structures could be arranged to differ (or be similar) in physical qualities (e.g. volume, mass, texture). And in case the source filament is looped - topological and geometric knot tools can be applied onto these structures.

Semantically: Knots can be used to code information. And sometimes for some reasons it is a preferred way.

Physical knots were famously used in South American cultures as a mnemonic system, called Khipu (or Quipu) [24, 5, 3]. Research shows, that the information was coded not only via the positioning of knots on a larger strand (geometry), but also with the types of knots used (topology) (Figure 6) [3].

Figure 6: Three types of knots in Khipu. From right to left: Long knot with 4 turns, Single knot (equal to a trefoil), Figure-eight knot (from [3])

At the same time, weaving could be thought of as a type of coding. The Jacquard machine, which was used to automate weaving patterns with looms, patented in 1804, used punch cards, and is recognized as one of precursors to modern computers [36]. Basically it was translating a code from cards into a sequences of braids. Quite obvious, that those braids could be read and translated back into cards. On the higher level of coding with knots and/or braids we can find descriptive and objective ornamental structures. For example lace in a shape of a flower or someone's face or a word.

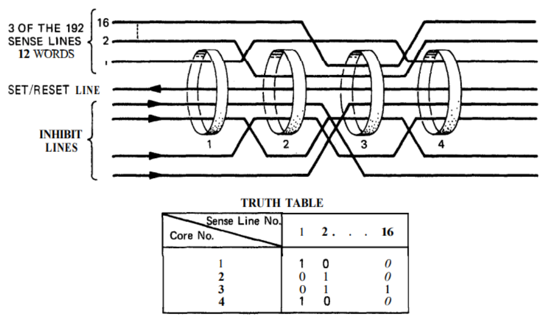

Figure 7: Core Rope Memory simplified diagram (from [17])

An interesting technique of weaving mnemonics for electronic hardware is a Core Rope Memory, implemented by NASA in Apollo Guidance Computer [17]. Although strictly speaking, we can not be sure, that the quality of crossings (over or under) had any significance in this device, it's simplified diagram is a braid projection (Figure 7).

2.2 Fields of Computer Music

Just for the convenience of this paper we arbitrarily divide Computer Music into these fields: Music Generation Systems (MGS), Spatiality (S), Digital Signal Processing (DSP), New Interfaces for Musical Expression (NIMEs).

2.3 Music Generation Systems

Numerous Music Generation Systems (MGS) exist [6, 41]. Given the time-based sequential nature of music, many MGS can be effectively represented as graphs, even accommodating polyphonic multichannel music [7].

For example Markov Chain algorithm - is a directed graph where vertices correspond to states and directed edges represent state transitions [8].

Knot projections - are planar graphs, where crossings become vertices and arcs represent strands.

Derived from knot diagrams, specific graph types, like medial graphs, add vertices at arc midpoints, with edges indicating strand crossings, preserving both topology and spatial relationships [28].

There are many connections between Graph Theory and Knot Theory [21, 15, 28].

We can see, that by using graphs in MGS, we sometimes use knot projections. What can be a knot-specific application?

2.3.1 Geometrically:

When considering applications of geometrical properties, we mostly will talk about knotted trajectories within parametric spaces. In the context of MGS, this translates to a compositional parametric space, with each dimension representing a gradient of musical change.

A knot shadow, depicting a tangled move in 2D space, can serve as a graphic score.

Extending to a knot diagram introduces an additional condition - information about crossings (two states) for other parameters.

A 3D knot manifests movements in three composition dimensions.

Notably, each 3D knot, in various spatial orientations, can produce multiple simultaneous projections, all looped. For instance, a 3D trefoil realization within a cube generates six shadows, one on each side, potentially serving as trajectories for distinct musical changes. Rotation of the knot induces a synchronized alteration across all six scores (Figure 8).

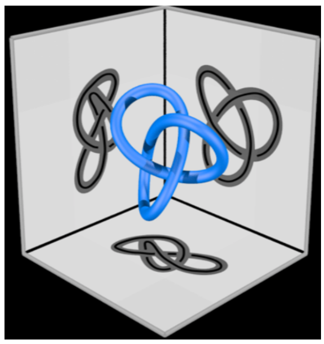

Figure 8: Three momentary shadows of a trefoil inside a half of a cube (from [9])

2.3.2 Topologically:

Achieving composition transitions between complex states while maintaining recognizable qualities can be accomplished through sequences of Reidemeister Moves. Graph-based algorithms, particularly those operating on intrinsically knotted graphs, offer identifiable knotted paths aligning with Knot Theory tools [15].

The multitude of invariants among the 350 million tabulated knots allows for diverse recordings of intrinsic knots within graphs, potentially enhancing memory organization in knot-related MGS through simple algebraic expressions [4].

2.3.3 Physically:

Imagine a rubber band with various knots: a large "monkey fist," a composite knot, lightweight "overhand" knots. Each knot becomes a distinct "rubber ball" connected elastically to the next. When dropped onto a resonating surface, this system creates partly-random sound sequences.

Untying the knots and forming three large "monkey fist" knots of equal volume from the same rubber band (Figure 9) yields a new system with differing core temporal and timbral characteristics, despite still generating partly-random sequences when dropped onto the same membrane.

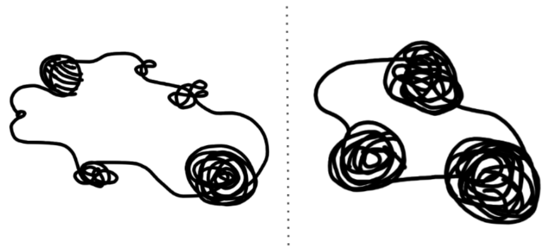

Figure 9: Two physical systems made from one filament

2.4 Spatiality

Space manipulation is crucial for computer music practitioners, with multi-layered spatial composition ranking as the third criterion in Karlheinz Stockhausen's Four Criteria of Electronic Music [37]. Various computation technologies, such as Ambisonics, Vector Base Amplitude Panning (VBAP), Dolby Atmos, and Wave Field Synthesis, aim to allow composers or performers to place virtual sound sources anywhere in space around the listener. Can knots be useful here?

2.4.1 Geometrically:

Knot computations involve calculating a curved trajectory in 3D, offering possibilities for moving virtual sound sources with tangled dynamics or static positioning along knotted lines. Projections can be useful when spatial systems use a single-level array of speakers, lacking Z-axis information.

Immersive capabilities of speaker-based systems often hinge on the number of speaker arrays, making them costly. Binaural modeling, delivered via headphones or ultra-directional close-range speakers, may prove more effective in convincingly moving virtual sound objects, especially in challenging scenarios like a room with two people.

Imagine a spatial composition in VR. A single virtual sound source moves around each listener within a shared VR space, following knotted trajectories. If listeners are dynamic, the sound source adapts, continuously recalculating knotted paths to accommodate their changing positions, including entries and exits. (Figure 10).

A proof-of-concept for this example was realised in Unity3D engine by scripting a parametric trefoil trajectory to a 3D object, while using a first person game controller and stereo audio render (Figure 11). We can see, that in VR it becomes possible for the user to walk into the knotted structure and listen to the virtual sound source (blue cube), constantly moving in a knotted path.

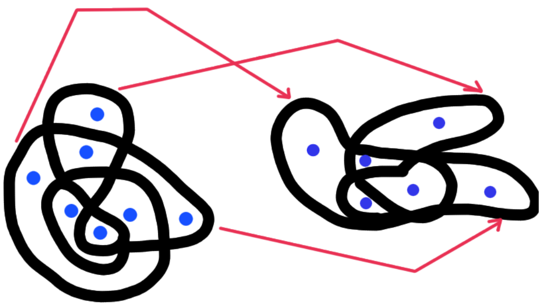

Figure 10: Knotted path (black) around listeners (blue) in dynamic change (red)

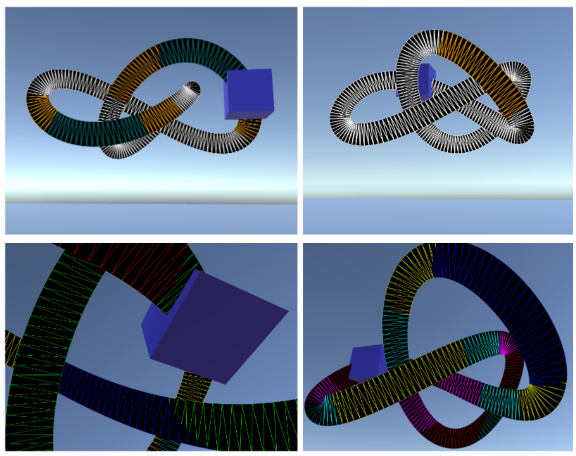

Figure 11: Trefoil trajectory in VR space. Using a first-person controller, we can walk inside the virtual space. Thus we walk to the sound object (cube), while it moves in a trefoil path first in front of us, and later - around our head.

2.5 Digital Signal Processing

A number of research exists in the intersection between signal processing and topology. Some of it explores representations of harmonic patterns in sound and visuals [31]. Other shows a rigorous work in mappings between synthesis and topological structures [14, 13, 12]. And some even model hardware (not necessarily digital) [32]. Still this area is largely new and waits deeper study, and knots in particular have not always been in a focus of attention.

As mentioned earlier - a mathematical knot is a tangled (sometimes to a level of knotting) circle in 3D. In DSP, circle usually represents a sinusoidal signal. Essl [14, 12] showed the possibility of synthesis by deforming oscillatory signals, topologically mapped on circles. Hence we can imagine knots as representation (or generation) of specific wave shapes from circles deformed into knots.

Another direct connection between knots and synthesis is a type of knots called Lissajous knots, which give Lissajous curves as their projections on any of three coordinate planes. It is proven, that all (3, q)-torus knots have Lissajous projections as well [22, 20]. Both Lissajous and Torus knots are defined by 3 sinusoidal parametric equations and can be translated as 3 sound oscillators.

2.5.1 *Geometrically:*

Here we would like to show an example of proof of concept for the Torus knot synthesizer (Figure 12 to Figure 15). It maps "p" and "q" variables as angular frequencies (usually "w") for 3 oscillators, identified earlier (*see equations (1)*). We took (q, p) values for the first five Torus knots from Adams, Hildebrand and Weeks [2]:

$$3_1 knot(3,2),\ 5_1 knot(5,2),\ 7_1 knot(7,2),$$

$$8_{19} knot(4,3),\ 9_1 knot(9,2).$$

The three oscillators are interpreted as sounding mixed (added) together. According to the equations (1), oscillators "x" and "y" use amplitude modulation ("r"), and oscillator "z" is inverted to represent negative amplitude (usually "A"). The instrument is realised as a Pure Data patch [30] and GEM (Graphic Environment for Multimedia, written by Mark Danks and now maintained by Johannes Zmölnig) is used for 3D visualisation.

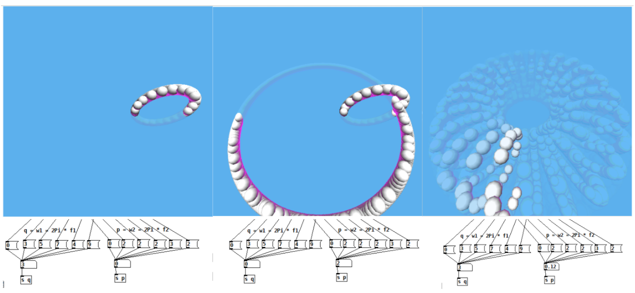

Figure 12: The start of a Torus forming seen in real-time along with frequencies of the oscillators raising: q=3 p=0 (left) ; q=0 p=2 (center) ; q=3 p=0.12 (right)

2.5.2 *Topologically:*

Several existing synthesis techniques were shown to be mapped on topological structures in [13, 12, 14]. This was done through the use of mathematical tool called "sheaves", understanding of which probably requires a deeper training

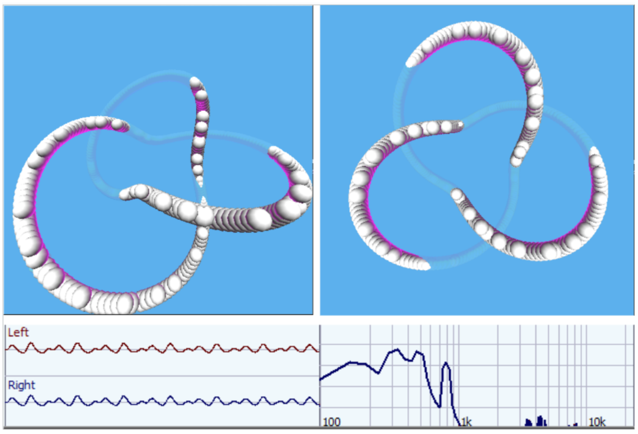

Figure 13: Two angles of view on a stable $3_1 knot(3,2)$ "$trefoil$" with corresponding waveform and spectrum

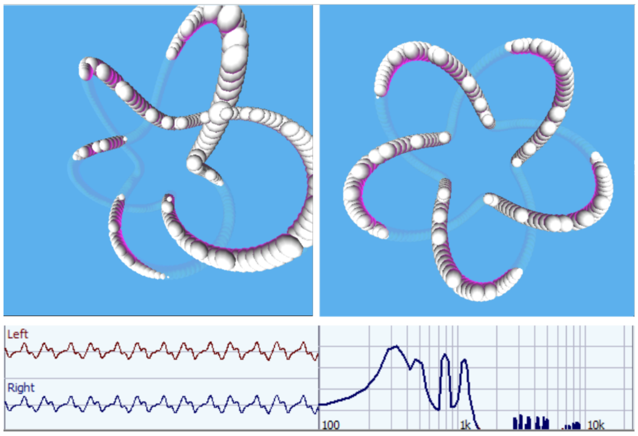

Figure 14: $5_1 knot(5,2)$

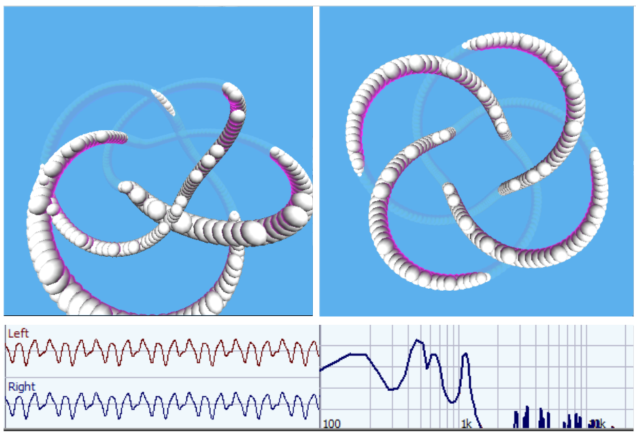

Figure 15: $8_{19} knot(4,3)$

in topology. One particular case (example 3 from [13]) describes a Frequency Modulation as a Torus knot projection on a plane.

2.5.3 Physically:

One interesting possibility lays in the use of knotted shapes in physical modelling.

Examples here could be:

1) a knotted pipe with valves in place of crossings

2) a dynamically knotted string, which could change it's thickness to become a knotted tubular bell

3) a model of the structure, which can alternate masses, proposed earlier as MGS (Figure 9)

2.6 NIMEs

We may say, that every example mentioned so far has an implicit interface possibility in itself. Especially as the starting point for our exploration - knot - is a tangible phenomena, which was first abstracted into images, and later to symbols. Nevertheless, some ideas could be instigated by thinking of performative or interactive interface as a starting point. Those perhaps would fit better into the current category.

2.6.1 Geometrically:

Some work in haptic exploration of mathematical knots (represented geometrically), is shown in [39, 40]. In particular in [40] we find a prototype for the display-based interface, where knots could be loaded and manipulated. And if previously such interfaces would imply modelling of physical properties of rope, here we see authors using tools from Knot Theory. Like the interaction, which is limited to the set of Reidemeister moves "resulting in a more fluid yet mathematically correct user experience with knots".

Another publication is more focused on haptic feedback during the exploration of virtual knots with touchscreen [39]. Researchers try to enrich user experience in haptic tracing of mathematical knot, which is represented on a flat screen (thus being a diagram), to the extent of "feeling" the 3D structure. Interestingly enough, sound tags are used here to distinguish "each over and under crossing".

These types of experimental interactive interfaces, could be mapped on various parameters of sound.

2.6.2 Topologically:

Examples of application of Knot Theory to the study of musical gestures is given in [25, 26]. The authors describe mathematical theory of musical gestures and generalize it to include knots and braids. This can be considered as a purely topological application, because knot theory features are used here for the tasks of recognition and classification - the initial purpose of the theory.

In case of interfaces, the natural field for such tasks - is browsing and menus. And this is a nice transition to semantic domain.

2.6.3 Semantically:

Multiple experimental research in psychology shows the superiority of pictograms in memorization and recognition [34, 16, 33]. As in computer music a practitioner has to constantly deal with very abstract sounds, challenge may be to find picture representations for collections of them. We can imagine a menu, where abstract sounds, or synthesizer patches could be assigned with knot diagrams.

In this case a topological equivalence can be useful, as we know, that every knot can be represented by infinite number of diagrams and even larger number of shadows.

For example, a 3-oscillator basic patch is represented by a trefoil diagram in it's simplest symmetrical form. Then every next alternation of this basic patch can be represented by every next diagram of a trefoil. At the same time, a basic patch of 4 oscillators would be assigned with a simplest symmetrical figure-eight knot diagram, and etc (Figure 16).

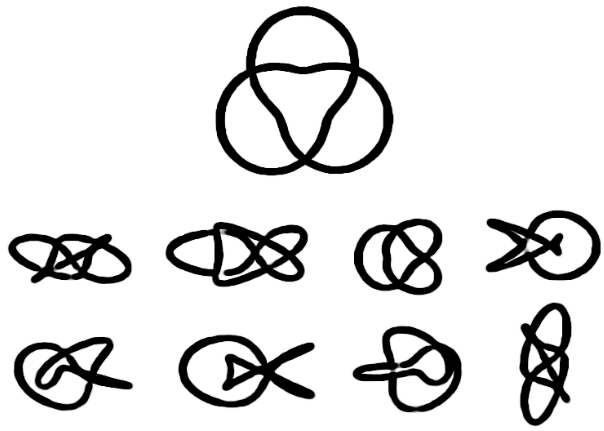

Figure 16: Possible synthesizer menu: basic trefoil (a synth patch) and 8 first shadows of it (subpatches)

2.6.4 Physically:

Finally, it might be useful to think of possibilities of using knots as physical controllers.

An artistic work has been done in this regards by Cadavid [5]. By implementing discussed earlier Khipu into a NIME, author not only used semantic and physical features of knots, but also put the whole project into a cultural and historical context, at the same time keeping it aesthetically beautiful and sonically engaging. Although knots were used here as electronic sensors, the project itself was bigger, than that. That is why, it actually does not explore the mechanical possibilities of knots as devices.

In the start we've mentioned that knots can be thought of as configurable machines. We can try to conceptualize a tangible haptic controller in a form of a rope or a cable, which would be able to change the functionality of software according to the different shapes of knots formed with it.

We can take Knot Theory tool - invariant - to track the shape of the knots. An invariant "number of crossings" can work in this case.

The user instruction for such controller can be:

1) tie a knot of your choice

2) press every crossing one by one for calibration

3) use loops of the knot as buttons or faders

Such controller can utilize mechanical advantages of ropes and knots. For example it can be wrapped around performer's body, or change shape and function depending on the number of players (Figure 17).

The crucial part of such device would be a design of a flexible sensor, which could register crossings and assign different parts of the tangled filament with different parameters.

That is why for the proof of concept of it we decided to prototype such sensor. It uses Velostat material, conductive foil, resistive rubber border in between the two and a protective cover (Figure 18). We show, that such sensor is

Figure 17: Three hypothetical different controllers from one rope-like source, thanks to topological invariant. Every other shape can trigger different mode, according to the number of crossings.

capable of registering a push in a form of two different resistance measurements from both ends (Figures 19, 20). This divides the filament in two parts. Obviously, a crossing of the rope (not a circle) requires a division in three. With the further engineering this might be improved.

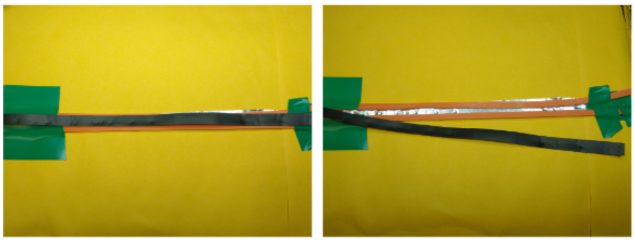

Figure 18: Inside the sensor: velostat, foil, rubber

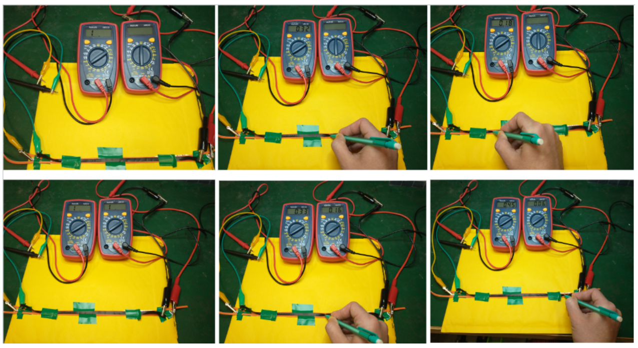

Figure 19: Test of push reading on a table. The two Ohmmeters show relative readings of a push on a horizontal surface. This can allow to divide the stripe in two parts in subsequent computation. Each part then can be assigned to control various parameters.

3. CONCLUSION

In this work we have explored briefly phenomena of knots. We took a mathematical Knot Theory for the start and derived two main domains of knots: geometrical and topological. After adding another two domains - physical and semantic - we have outlined several examples of applications of knots in four areas of computer music. We showed conceptual examples for Music Generation systems in geometrical, topological and physical domains. We developed proof of concept prototype for Spatial sound application in VR space, using geometrical calculation of a Trefoil Knot.

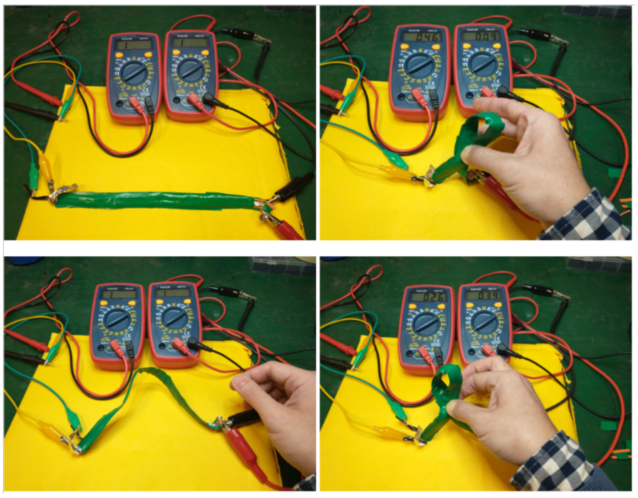

Figure 20: Test of crossing reading. We can see, that the crossing is registered as the same kind of simultaneous relative reading of two resistances as in the previous picture. The future computation should invoke logic to make it possible to divide the stripe in three parts from this interaction.

We made a Torus Knot Synthesizer as an example of a simple DSP application of the same geometrical features but now computing several different torus knots. Finally we illustrated the possible use of topological domain in menu design and tested a setup of a real physical controller, which could be turned into a knot-registering sensor.

In some cases (like physical domain for Spatiality) - we could not think of a good enough examples. In others (like a Topological domain for Spatiality) - it would be repetition, because a Geometrical examples already involve use of topological tools. And in case of Semantic domain for any field, other than NIMEs - the examples would be too obvious (a knot-based coding language for controlling MGS, or spatial distribution or synthesis). Perhaps more interesting and useful applications could be realised, and this we leave for future studies.

It is important to state that this work is a short outline of some of the possibilities and does not aspire to be exhaustive. The main purpose of this paper is to instigate interest in knotted structures, facilitate their exploration in music technologies and perhaps contribute to a dialog about it.

4. ACKNOWLEDGMENTS

This work was possible thanks to consultations with Tamara Smyth, Georg Essl and Justin Roberts.

5. ETHICAL STANDARDS

This research was done utilizing time for individual PHD studies, without additional funding from any source, and did not involve experiments with human or animal participation.

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